4.3 Eigenvalues and Eigenvectors of $n\times n$ matrices

Def. $A_{n\times n}$ then
λ in $\mathbb{R}$ is an eigenvalue of A and $\vec{x} \neq \vec{0}$ in $\mathbb{R}^n$ is a corresponding eigenvector if

$$\boxed{A\vec{x} = \lambda\vec{x}} \qquad (1.1)$$

Theorem. $A_{n\times n}$ and $\lambda \in \mathbb{R}$

λ is an eigenvalue of A if and only if

$$\boxed{(A-\lambda I)\vec{x} = \vec{0}} \qquad (1.2)$$

has nontrivial solutions

Corollary $A_{n\times n}$ and $\lambda \in \mathbb{R}$

λ is an eigenvalue of A if and only if

$$\boxed{\det(A-\lambda I) = 0} \qquad (1.3)$$

Corollary $A_{n\times n}$

$\lambda = 0$ is an eigenvalue of A if and only if A is not invertible.

Def. The set of all solutions of (1.2) is called the Eigenspace of A corresponding to λ

Notice that the eigenspace of A is

$$\text{null}(A - \lambda I)$$

Also, $\text{null}(A - \lambda I)$ contains $\vec{x} = \vec{0}$ which is not an eigenvector.

Exercise Find the eigenvalues of A and basis for the eigenspaces for

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$ we need to find λ's first

$$|A - \lambda I| = 0 \iff \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

Using Cofactor exponsion along 1^{st} row

$$\det(A - \lambda I) = |A - \lambda I| = (\lambda - 1)(\lambda - 3)(\lambda - 2)$$

In more detail,

$$|A-\lambda I|=0 \iff \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

or employing Cofactor expansion along 1^{st} row:

$$0=(4-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0\begin{vmatrix} -2 & 0 \\ -2 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} -2 & 1-\lambda \\ -2 & 0 \end{vmatrix}$$

$$= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda)$$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right] = (1-\lambda)\left[(\lambda-1)(\lambda-4)+2\right]$$

$$= -(\lambda-1)\left[\lambda^2-5\lambda+6\right] = -(\lambda-1)(\lambda-3)(\lambda-2)$$

$$= (1-\lambda)(2-\lambda)(3-\lambda).$$

So, $|A-\lambda I|=0$ is represented by

$$P(\lambda)=0$$

where $P(\lambda) = -(\lambda-1)(\lambda-3)(\lambda-2)$

or $\boxed{P(\lambda) = -\lambda^3+6\lambda^2-11\lambda+6}$ This is called the Characteristic polynomial of the matrix A

In case that you arrive to the characteristic equation

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0} \qquad (*)$$

How do you obtain the eigenvalues?

Remark: Given a polynomial: $\boxed{\lambda^n + c_1\lambda^{n-1} + \dots + c_n = 0}$

with integer coefficients, all integer solutions (if any) must be a divisor of the constant term C_n.

In our case, equation $(*)$ has $C_n = 6 \Rightarrow$ Divisors $\pm 1, \pm 2, \pm 3$

In particular,

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 0 \checkmark \quad \lambda_1 = 1$$

$$\Rightarrow \begin{array}{l|l} \cancel{\lambda^3} - 6\lambda^2 + 11\lambda - 6 & \lambda - 1 \\ \hline -\cancel{\lambda^3} + \lambda^2 & \lambda^2 - 5\lambda + 6 \\ \quad -5\cancel{\lambda^2} + 11\lambda - 6 & \\ \quad +\cancel{5\lambda^2} - 5\lambda & \\ \qquad 6\lambda - 6 & \\ \qquad -6\lambda + 6 & \\ \qquad 0 & \end{array}$$

then,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$$

$$= (\lambda - 1)(\lambda - 3)(\lambda - 2) \checkmark$$

Then, the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \text{ and } \lambda_3 = 3$$

To find eigenvectors for each λ, we need to substitute $\lambda_i^{(i=1,2,3)}$ into equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

$$\lambda_1 = 1 \Rightarrow (A - I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} 4-1 & 0 & 1 \\ -2 & 1-1 & 0 \\ -2 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{2}{3}u_3 = 0 \Rightarrow \boxed{u_3 = 0}, \qquad u_1 = -\tfrac{1}{3}u_3 = 0$$

$$\Rightarrow \boxed{u_1 = 0}, \ u_2 \text{ free}$$

then.

$$\vec{u} = \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}, \ t \neq 0.$$

The eigenspace for $\lambda_1 = 1$

$$E_{\lambda_1=1} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{u}},\ t \in \mathbb{R} \right\}$$

Similarly,

$$E_{\lambda_2=2} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \underbrace{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}}_{=\vec{v}},\ t \in \mathbb{R} \right\}$$

$= \vec{v}$ eigenvector corresp. to $\lambda_2 = 2$

$$E_{\lambda_3=3} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ t \in \mathbb{R} \right\}$$

eigenvector corresp. to $\lambda_3 = 3$.

Check this answer.

$\boxed{\text{Relationship of det } A}$ with $\boxed{\text{eigenvalues of } A.}$

Given
$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
find $\det A$.

Choosing the 2^{nd} column

$$\det A = 1(-1)^{2+2} \begin{vmatrix} 4 & 1 \\ -2 & 1 \end{vmatrix} = 1(6) = 6$$

So, $\boxed{\det A = 6 = 1.2.3 = \lambda_1 \lambda_2 \lambda_3}$

This is true in general. as our next theorem states

<u>Theorem.</u> 1) $A_{n \times n}$ matrix

2) A has n real eigenvalues $\lambda_1, .., \lambda_n$ including repetitions

then

$$\boxed{\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n}$$

<u>Proof.</u> The characteristic polynomial of A is

$$\boxed{\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda)} \quad (5.1)$$

Once, it has been factorized using its roots. Now, these roots are also the eigenvalues of A.

If in (5.1) we substitute $\underline{\lambda = 0}$

$$\det A = \det(A - \overset{=0}{\lambda} I) = (\lambda_1 - 0)(\lambda_2 - 0) \ldots (\lambda_n - 0)$$
$$= \lambda_1 \cdot \lambda_2 \ldots \lambda_n$$

or $\boxed{\det A = \lambda_1 \ldots \lambda_n}$

Assume $A \sim U$ and U is in echelon form

We want to know the relationship between the eigenvalues of A and the eigenvalues of U.

In our previous example

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \underset{\substack{R_2 + R_1/2 \\ R_3 + R_1/2}}{\sim} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 3/2 \end{bmatrix} = U$$

Because U is upper triangular its eigenvalues are $\boxed{\alpha_1 = 4, \ \alpha_2 = 1, \ \alpha_3 = 3/2}$ U eigenvalues

Clearly, different than $\boxed{\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3}$ A eigenvalues

which are the eigenvalues of A.

However, $\boxed{\det A = 1.2.3 = 6 = 4.1.3/2 = \det U}$

<u>Conclusion</u>: a) Row operations change the eigenvalues of the original matrix

b) If row operations only involves replacements then $\det A = \det U$.

Which are the eigenvalues of an upper triangular matrix. ?

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 2-\lambda & -5 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (3-\lambda)(2-\lambda)(7-\lambda) = 0$$

$\Rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = 7$ are its eigenvalues.

thm $A_{n \times n}$

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Can $\lambda = 0$ be an eigenvalue for $A_{n\times n}$?

Consider

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

then $\boxed{\lambda_1 = 3,\ \lambda_2 = 0,\ \lambda_3 = 1 \text{ are the eigenvalues of } A.}$

If we replace $\lambda_2 = 0$ into $\det(A - \lambda I)$

we obtain $\det(A) = \det(A - 0I) = 0$

It means A is noninvertible. It means if $\lambda = 0$ is an eigenvalue of A, then A is noninvertible.
It's easy to prove the other way (A non-inv. $\Rightarrow \lambda = 0$ is Eigenvalue)
Then, we have the equivalent statement:

$$\boxed{A \text{ is invertible} \iff \lambda = 0 \text{ is not an eigenvalue of } A.}$$

Compute Eigenvalues, Eigenvectors, a basis for the eigenspaces and the $\boxed{\text{algebraic and geometric}}$ multiplicity.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \det(A - \lambda I) = 0$$

$$\Updownarrow$$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

Cofactor expansion along 1st row:

$$0 = (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda^2-1-1) - 2(\lambda-1) = 0$$

$$\Leftrightarrow (\lambda-1)\left[\lambda^2-2+2\right] = 0 \Leftrightarrow (\lambda-1)\lambda^2 = 0$$

$\Rightarrow$ $\lambda_1 = 0$ (twice) algebraic multiplicity 2.

and $\lambda_2 = 1$ algebraic multiplicity 1.

Eigenvectors associated to $\lambda_1 = 0$.

Same as nontrivial Solns for homogeneous System:

$$A\vec{x} = \vec{0} \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 = -x_3 \\ x_1 = -2x_2 = 2x_3 \end{array}$$

Eigenvectors:

$$\vec{X} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$E_{\lambda=0} = \left\{ \vec{x} \in \mathbb{R}^3 : \ \vec{X} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \right\}$$

Then geom. multiplicity of $\lambda_1 = 0$ is 1.
Because basis consists of only 1 vector for Eigenspace
$E_{\lambda=0}$.

Similarly for $\lambda_2 = 1$

$$(A - I)\vec{x} = 0 \iff \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 = 0 \Rightarrow \boxed{x_2 = 0}$$

$$-x_1 = -x_3 + 2\overset{=0}{x_2} \Rightarrow \boxed{x_1 = x_3}$$

$$\vec{x} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

then

$$E_{\lambda=1} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

Geometric multiplicity 1.

Eigenvectors Corresponding to different eigenvalues are linearly independent.

Given $A_{n\times n}$ and $\lambda_1 \neq \lambda_2$ two eigenvalues of A, and $\vec{v}_1 \neq 0$ and $\vec{v}_2 \neq 0$ their corresponding eigenvectors. The following holds:

The set $\{\vec{v}_1, \vec{v}_2\}$ is lin indep.

Proof.-

w.t.p.

If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \Rightarrow c_1 = 0, \ c_2 = 0.$

Let $\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}}$ (1)

Multiplying by A

$$A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = A\vec{0} = \vec{0}$$

or $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0} \Leftrightarrow \boxed{c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}}$ (2)

If we also multiply (1) by λ_1 and subtract (2) from it

$$\begin{array}{r} c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0} \\ -\left[c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}\right] \\ \hline c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0} \end{array}$$

$\Rightarrow c_2 = 0$ since $\lambda_2 \neq \lambda_1$ and $\vec{v}_2 \neq 0$.

Similarly we can show that $c_1 = 0$, or better go to (1) and substitute $c_2 = 0$.

This result can be extended by induction to any number of distinct eigenvalues:

<u>Theorem</u>: $A_{n\times n}$ matrix

If $\lambda_1, \lambda_2 .., \lambda_m$ are distinct eigenvalues of A with corresponding eigenvectors $\vec{v}_1, \vec{v}_2 .., \vec{v}_m$, then $\vec{v}_1 .., \vec{v}_m$ are linearly independent.